Assignment 6

This homework is due Friday March 6.

There are total 45 points in this assignment. 40 points is considered 100%. If you go over 40 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should exhibit your work and contain full proofs. Bare answers will not earn you much.

This assignment covers Sections 3.2, 3.3, 4.1, partially 4.3 of Textbook.

(1) [5pt] Find what form Cauchy–Riemann equations take in polar coordinates, following the procedure below.
   (a) Let $f(z) = u(x, y) + iv(x, y)$. Plug in $x = r\cos\theta$ and $y = r\sin\theta$ and find partial derivatives of u, v w.r.t to r and θ using chain rule.
   (b) Use Cauchy–Riemann equations for u_x, u_y, v_x, v_y to establish relation between u_r and v_θ, and between u_θ and v_r.

(2) [5pt] Establish whether the following functions are analytic in the domain $r > 0$, $-\pi < \theta < \pi$ by checking Cauchy–Riemann conditions in polar form.
   (a) $f(z) = \sqrt[3]{z}$ (principal cubic root function), that is $f(z) = f(re^{i\theta}) = r^{\frac{1}{3}}(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3})$.
   (b) $f(z) = (\ln r)^2 - \theta^2 + i2\theta\ln r$.

(3) [5pt] Establish which of the following functions are harmonic.
   (a) $u(x, y) = e^x \cos y$.
   (b) $u(x, y) = \arctan\frac{y}{x}$, $x \neq 0$.
   (c) $u(x, y) = x^2 + y^2$.
   (d) $u(x, y) = \ln(x^2 + y^2)$, $(x, y) \neq (0, 0)$.

(4) [10pt] For a given $u(x, y)$ find $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$ is analytic. If no such v exists, explain why.
   (a) $u(x, y) = x^3 - y^3$.
   (b) $u(x, y) = 4x^3y - 4xy^3$.
   (c) $u(x, y) = \sin y \sinh x$.
   (d) $u(x, y) = e^y \sin x$.

(5) [5pt] Find the following limits. (*Hint* for (b): $\frac{n+i^n}{n} = \frac{1+i^n/n}{1}$. In (c), (d) similarly divide by the largest term in the numerator and the denominator.)
   (a) $\lim\limits_{n\to\infty} \left(\frac{1}{2} + \frac{i}{5}\right)^n$.
   (b) $\lim\limits_{n\to\infty} \frac{n+i^n}{n}$.
   (c) $\lim\limits_{n\to\infty} \frac{n^2+i2^n}{2^n}$.
   (d) $\lim\limits_{n\to\infty} \frac{(n+i)(1+ni)}{n^2}$.

(6) [10pt]
   (a) $\lim\limits_{n\to\infty} \left(\frac{1+i}{\sqrt{2}}\right)^n$ exist? Why?
   (b) Show that $\sum\limits_{n=0}^{\infty} \left(\frac{1}{n+1+i} - \frac{1}{n+i}\right) = i$.
   (c) Does $\sum\limits_{n=1}^{\infty} \frac{i^n}{n}$ converge? Why?

(7) [5pt] Use the ratio test to establish that the following series converge.
   (a) $\sum\limits_{n=0}^{\infty} \frac{(1+i)^n}{n2^n}$.   (b) $\sum\limits_{n=1}^{\infty} \frac{(3+4i)^n}{n!}$.